Conditional Probability

When additional information is known that will affect the probability of an outcome, we calculate what is called a conditional probability, denoted $P(A|B)$, using

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

where the notation $P(A|B)$ is read "the probability of the event A, given that the event B has occurred," or more succinctly, "the probability of A, given B."

Note that once B is known then the event set for A is restricted to those outcomes in the set $(A \cap B)$. Furthermore, the outcome set has been reduced from the set S to the set B. Hence,

$$\boxed{P(A|B) = \frac{n(A \cap B)}{n(B)}}$$

Example 1: What is the probability of rolling a sum greater than 7 with two dice if it is known that the first die rolled is a 3?

Solution: Let event A be a sum greater than 7 and event B be the even that the first die rolled is a 3.

$B = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$

$\therefore P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

$\because P(B) = \frac{6}{36} = \frac{1}{6}$

$\because P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(1/18)}{(1/6)} = \frac{1}{18} \times \frac{6}{1} = \frac{1}{3}$

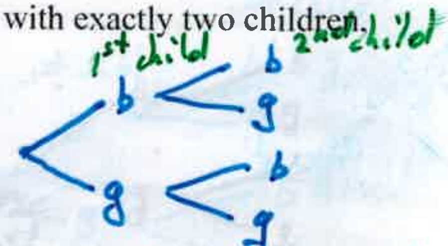
OR $P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{6} = \frac{1}{3}$

$\therefore$ There is a prob. of $\frac{1}{3}$ of rolling a sum greater than 7 if it is known that the first die rolled is a 3.

Example 2: If a family is chosen at random from the set of all families with exactly two children, find the probability that

a) the family has two boys if it is known that the one child is a boy.

b) the family has two boys if it is known that the first child is a boy.



$S = \{bb, bg, gb, gg\}$

$n(S) = 4$

Sol: a) Let event A be the family has two boys and let event B be that the family has at least one boy.

$B = \{bb, bg, gb\}$, $A \cap B = \{bb\}$

$n(B) = 3$, $n(A \cap B) = 1$

$\therefore P(B) = \frac{3}{4}$

$\therefore P(A \cap B) = \frac{1}{4} \leftarrow \frac{n(A \cap B)}{n(S)}$

$\because P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(1/4)}{(3/4)} = \frac{1}{\cancel{4}} \times \frac{\cancel{4}}{3} = \frac{1}{3}$

b) Let B be the event that the first child is a boy.

$B = \{bb, bg\}$, $n(B) = 2$

$P(B) = \frac{2}{4} = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$

$\because P(A|B) = \frac{(1/4)}{(1/2)} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}$

Rearrange the equation for conditional probability for $P(A \cap B)$, we obtain the multiplication law for conditional probability.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad / \times P(B)$$

$$P(B)\,P(A|B) = P(A \cap B)$$

$$\boxed{P(A \cap B) = P(B)\,P(A|B)}$$

This is the probability of events A and B occurring, given that event A has occurred.

Example 3: The probability that Epsilon will go to Eastern University is 1/5. The probability that he will go to another university is 1/2. If Epsilon goes to Eastern, the probability that his girlfriend Gamma will follow him and go to Eastern is 3/4. What is the probability that both Epsilon and Gamma attend Eastern University?

Sol: Let A be the event that Epsilon goes to EU and let B be the event that Gamma goes to EU.

$P(A) = \frac{1}{5}$

$P(B|A) = \frac{3}{4}$

$$P(B \cap A) = P(A)\,P(B|A) = \frac{1}{5} \times \frac{3}{4} = \frac{3}{20} = 0.15 = 15\%$$

∴ There is a prob. of 15% that both Epsilon and Gamma attend Eastern University.

Example 4: What is the probability of drawing two jacks in a row from a well-shuffled deck of 52 playing cards? The first card drawn is not replaced.

Sol: Let A be the event that the first card drawn is a jack. Let B be the event that the second card drawn is a jack

$P(A) = \frac{4}{52} = \frac{1}{13}$

$P(B|A) = \frac{3}{51}$

$$\therefore P(B \cap A) = P(A)\,P(B|A) = \frac{1}{13} \times \frac{3}{51} = \frac{3}{663} = \frac{1}{221} \doteq 0.0045 \doteq 0.45\%$$

∴ There is a prob. of approx. 0.5% of drawing two jacks in a row.

Homework: Pg. 235: #1,2,4,6-8,9a-e,10,14,WS