Solve Problems Involving Exponential Growth

Doubling – When the base of an exponential relation is 2, the relation is describing a doubling.

$$A = A_0(2)^{\frac{t}{d}}$$

A is the final amount, A_0 is the initial amount, t is the elapsed time, and d is the doubling time
The base is 2 because the amount is doubling

Example 1: The population, P, of penguins in a certain region of Antarctica can be modelled by the relation $P = P_0(2)^{\frac{t}{90}}$, where t is the time, measured in months and P_0 is the initial number of penguins.

a) What does the value 90 represent in this formula?
b) If there are 800 penguins in the region today, how many will there be in 2 years?
c) How long would it take for the number of penguins to reach 20000?

Ans:
a) 90 represents the number of months it takes for the penguins to double.

b) $t = 2$ years $= 24$ months and $P_0 = 800$

-Sub. $t = 24$ and $P_0 = 800$ into $P = P_0(2)^{\frac{t}{90}}$.

$P = 800(2)^{\frac{24}{90}}$
$\doteq 962$

∴ In 2 years, there will be approximately 962 penguins in the region.

c) $P = 20000$ $t = ?$

-Substitute $P = 20000$ into $P = 800(2)^{\frac{t}{90}}$ and simplify.

$20000 = 800(2)^{\frac{t}{90}}$

$\frac{20000}{800} = \frac{\cancel{800}(2)^{\frac{t}{90}}}{\cancel{800}}$

$\boxed{(2)^{\frac{t}{90}} = 25}$ ← solve the exponential equation

Algebraically

$\log(2)^{\frac{t}{90}} = \log 25$

$\frac{t}{90} \times \log 2 = \log 25$ $/\times 90$

$t\log 2 = 90\log 25$ $/\div \log 2$

$t = \frac{90\log 25}{\log 2}$

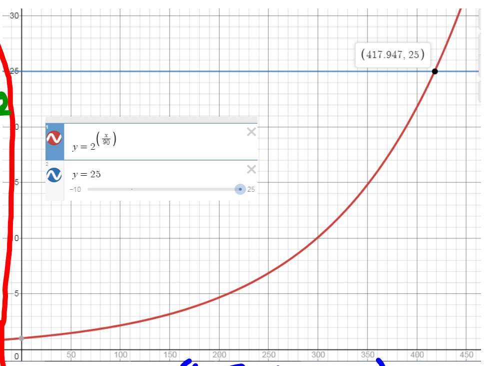
$t = 417.9$

$\boxed{t \doteq 418}$

∴ It will takes approx. 418 months for the population of penguins to reach 20000.

Graphically

$y_1 = 2^{\frac{t}{90}}$ and $y_2 = 25$



P.O.I. $= (417.947, 25)$

Example 2: Certain plant cells double every 40 minutes under controlled conditions. If there 700 cells initially, how long would it take for the bacteria to grow to 11200?

Ans: $A = A_0(2)^{\frac{t}{d}}$

$A_0 = 700$
$A = 11200$
$d = 40$

Substitute A, A_0 and d

$(*)\ldots$ $\boxed{11200 = 700(2)^{\frac{t}{40}}}$

- Simplify $(*)$

$$\frac{11200}{700} = \frac{\cancel{700}(2)^{\frac{t}{40}}}{\cancel{700}}$$

$$\boxed{16 = (2)^{\frac{t}{40}}} \leftarrow \text{exponential equation}$$

- Write 16 as a power with base 2

$$2^4 = 2^{\frac{t}{40}}$$

$$4 = \frac{t}{40} \leftarrow \text{equate the exponents}$$

$$4 \times 40 = t$$

$$\therefore \boxed{t = 160}$$

Using a table

t	A
0	700
40	1400
80	2800
120	5600
160	11200

∴ It will take 160 minutes for the bacteria to reach 11200.

Example 3: The world's population in 1980 was about 4.5 billion. Suppose the population increased at a rate of 2% per year since then.
a) Write an exponential relation to model the world's population.
b) Use your model to predict the world's population in 2019.
c) What was the population in 1970?
d) According to this model, in what year will the population reach 10.5 billion?

Ans:
a) $A = A_0 (b)^t$

$A_0 = 4.5$ ← initial population
t - time in years
A - final population
b - growth factor

$b = 1 + 0.02$
$= 1.02$

$\boxed{A = 4.5(1.02)^t}$ ... (**)

b) Substitute $t = 2019 - 1980 = 39$ into (**)

$A = 4.5(1.02)^{39}$
$\doteq 9.74$

According to this model, world's population is estimated to be approximately 9.74 billion in 2019 (this model clearly overpredicts the population)

c) $t = 1970 - 1980 = -10$

- Sub. $t = -10$ into (**)

$A = 4.5(1.02)^{-10}$
$= \frac{4.5}{(1.02)^{10}}$
$\doteq 3.69$

∴ World's population in 1970 was approximately 3.69 billion.

d) $A = 10.5$

Sub. $A = 10.5$ into (**) and simplify

$A = 4.5(1.02)^t$
$10.5 = 4.5(1.02)^t$
$\frac{10.5}{4.5} = \frac{\cancel{4.5}(1.02)^t}{\cancel{4.5}}$
$\boxed{2.33 = (1.02)^t}$

$t = \frac{\log 2.33}{\log 1.02}$
$\boxed{t \doteq 42.7}$

∴ World's population would reach 10.5 in $1980 + 42.7 = 2022.7$ (2022) according to this model.

Homework: Pg. 394: #17, 18, 21