Solve Problems Involving Exponential Decay

Exponential relations and their graphs can be used to solve problems in science, medicine, and finance. The ability to analyze data and model it using exponential relations is important for making accurate predictions.

Half-life: When the base of an exponential relation is 1/2, the relation is describing half-life.

Half-life is the time it takes for a quantity to decay or be reduced to half its initial amount.

Exponential Equations

$y = ab^x$ — a is the initial value, b is the growth/decay factor; b>1 models growth, 0<b<1 models decay

$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$ — A is the final amount, A_0 is the initial amount, t is the elapsed time, and h is the half life. The base is 0.5 because the amount is being reduced to half its original amount.

Nov 20-9:23 AM

Example 1: The remaining mass, M, in milligrams, of a drug in a person's bloodstream after t hours is modelled by the relation

$$M = 500\left(\frac{1}{2}\right)^{\frac{t}{2}}$$

a) What is the half-life of the drug?
b) What mass of the drug is in the bloodstream after 4 h?
c) How long would it take to reduce the amount of drug in the bloodstream to 62.5 mg?

Ans: $M = 500\left(\frac{1}{2}\right)^{\frac{t}{2}}$ $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$

a) The half-life of the drug is 2 hours.

b) $t=4$, so substitute $t=4$ into $M=500\left(\frac{1}{2}\right)^{\frac{t}{2}}$.

$M = 500(0.5)^{\frac{4}{2}}$
$= 500(0.5)^2$
$= 125$

∴ 125 milligrams of the drug is remaining in the blood stream after 4 hours.

c) $M = 62.5$
Substitute $M=62.5$ into $M=500\left(\frac{1}{2}\right)^{\frac{t}{2}}$ and solve for t.

$62.5 = 500\left(\frac{1}{2}\right)^{\frac{t}{2}}$

$\frac{62.5}{500} = \frac{500(0.5)^{\frac{t}{2}}}{500}$

$\boxed{0.125 = 0.5^{\frac{t}{2}}}$ ← exponential equation

-Write 0.125 as a power with base 0.5.

So, $(0.5)^3 = (0.5)^{\frac{t}{2}}$ ← $0.5 \times 0.5 \times 0.5 = 0.125$

$\frac{3}{1} = \frac{t}{2}$

$6 = t$

$\boxed{t=6}$

OR

$0.125 = (0.5)^{\frac{t}{2}}$

$\log 0.125 = \log (0.5)^{\frac{t}{2}}$

$\log 0.125 = \frac{t}{2}\log 0.5$ / ×2

$2\log 0.125 = t\log 0.5$ / ÷ log 0.5

$\frac{2\log 0.125}{\log 0.5} = \frac{t\cancel{\log 0.5}}{\cancel{\log 0.5}}$

$t = \frac{2\log 0.125}{\log 0.5}$

$\boxed{t = 6}$

Using a table

M	t
500	0
250	2
125	4
62.5	6

∴ It would take 6 hours for the drug to be reduced to 62.5 milligrams.

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Example 2: An isotope of sodium, Na-24, has a half-life of 15 h. If a sample contains 3.6 units of Na-24, then
a) Write an exponential function that describes this situation.
b) How long would it take for the amount to be reduced to 0.3 units of Na-24?

Ans:
a) $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$

$A_0 = 3.6 \leftarrow$ initial amount
$h = 15 \leftarrow$ half-life
A – final amount
t – time

$\boxed{A = 3.6\left(\frac{1}{2}\right)^{\frac{t}{15}}}$

b) $A = 0.3$
- Substitute $A = 0.3$ into $A = 3.6\left(\frac{1}{2}\right)^{\frac{t}{15}}$

$0.3 = 3.6(0.5)^{\frac{t}{15}} \leftarrow$ simplify

$\frac{0.3}{3.6} = \frac{\cancel{3.6}(0.5)^{\frac{t}{15}}}{\cancel{3.6}}$

$0.083 = (0.5)^{\frac{t}{15}}$

$\boxed{(0.5)^{\frac{t}{15}} = 0.083}$

Solve $(0.5)^{\frac{t}{15}} = 0.083$ algebraically

$\log(0.5)^{\frac{t}{15}} = \log 0.083$

$\left(\frac{t}{15}\right)\log 0.5 = \log 0.083$

$\cancel{15}\times\left(\frac{t}{\cancel{15}}\times\log 0.5\right) = 15\times\log 0.083$ / $\times 15$

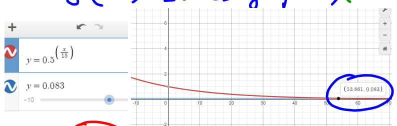
$t\log 0.5 = 15\times\log 0.083$

$\frac{t\cancel{\log 0.5}}{\cancel{\log 0.5}} = \frac{15\log 0.083}{\log 0.5}$ / $\div \log 0.5$

$t = \frac{15\log 0.083}{\log 0.5}$

$\boxed{t \doteq 53.9}$

- Solving $(0.5)^{\frac{t}{15}} = 0.083$ graphically (desmos.com)



P.O.I. = (53.861, 0.083)

So, it would take approx. 53.9 hours for the amount of Na-24 to be reduced to 0.3 units.

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Example 3: A new car depreciates at a rate of 30% per year. The purchase price was \$32 600.
a) Write an exponential function to model the depreciated value of the car.
b) What is the value of the car after 4 years?

Ans:
a) $y = a(b)^x$

$a = 32600 \leftarrow$ initial amount
x – time in years
y – final value

$\boxed{\begin{aligned} b &= 1 - 0.3 \\ &= 0.7 \end{aligned}}$
↑ decay factor

$\therefore \boxed{y = 32600(0.7)^x}$

b) Substitute $x = 4$ into $y = 32600(0.7)^x$.

$y = 32600(0.7)^4$
$\doteq 7827.26$

$\therefore$ The value of the car is \$7827.26 after 4 years.

Homework: Pg. 392: #12,14, 15, 16

May 11-11:37 AM